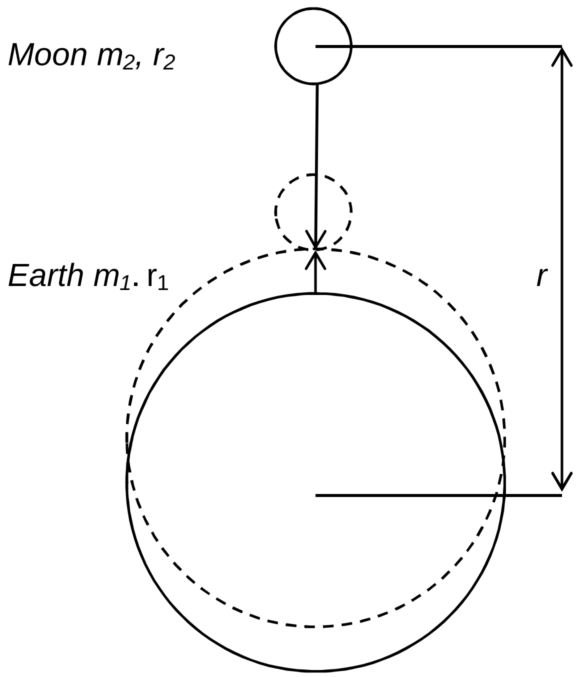
3. The effect of gravitation

When two masses move towards each other due to gravitation, the energy needed comes directly from these masses. The velocities that both masses are imparted is to be calculated, based on the example of the earth and the moon as idealized spheres, when they start to move towards each other from their mean real distance out of a theoretical state of rest until they collide.



Legend

$m_1 = m_{Er}$ =	5.977×10^{24} kg
$m_2 = m_{Mo}$ =	7.352×10^{22} kg
$r_1 = r_{Er}$ =	6.378×10^{6} m
$r_2 = r_{Mo}$ =	1.738×10^{6} m
r =	3.844×10^{8} m
h =	*Surface distance*
γ=	6.673×10^{-11} Nm²/kg²
Index a	*Rest position*
Index b	*State of motion during collision*

Fig. 03-01: Two-mass arrangement (not to scale)

The work that is performed until the two masses touch can be calculated as follows: [8]

$$E_{grav} = \int_{r_1+r_2}^{r} F_{grav} ds = \int_{r_1+r_2}^{r} \gamma \frac{m_1 m_2}{s^2} ds = -\gamma \frac{m_1 m_2}{s} \Big|_{r_1+r_2}^{r}$$

$$E_{grav} = -\gamma m_1 m_2 \left(\frac{1}{r} - \frac{1}{r_1+r_2}\right) = \gamma m_1 m_2 \left(\frac{1}{r_1+r_2} - \frac{1}{r}\right)$$

In addition, energy conservation must apply for both the accelerated masses, taking gravitation and relativistic mass changes on impact into account:

$$E_{pot} = E_{grav} = E_{kin1} + E_{kin2} = \Delta m_1 c^2 + \Delta m_2 c^2 \qquad [03\text{-}01]$$

The following can be written:

$$\gamma m_{1a} m_{2a} \left(\frac{1}{r_1+r_2} - \frac{1}{r}\right) = (m_{1b} - m_{1a})c^2 + (m_{2b} - m_{2a})c^2 \qquad [03\text{-}02]$$

[8] E.g. Dieter Meschede, Gerthsen Physik, P. 49, Springer-Verlag Heidelberg, 2002, ISBN 3-540-42024-X

The formula can then be expressed as follows in relativistic notation:

$$\gamma m_{1a} m_{2a}\left(\frac{1}{r_1+r_2}-\frac{1}{r}\right)=\frac{m_{1a}c^2}{\sqrt{1-\frac{v_{1b}{}^2}{c^2}}}-m_{1a}c^2+\frac{m_{2a}c^2}{\sqrt{1-\frac{v_{2b}{}^2}{c^2}}}-m_{2a}c^2$$

For velocities that are well below the speed of light, this equation can be put into the following form using the well-known series expansion :

$$\gamma m_{1a} m_{2a}\left(\frac{1}{r_1+r_2}-\frac{1}{r}\right)=\frac{m_{1a}}{2}v_{1b}{}^2+\frac{m_{2a}}{2}v_{2b}{}^2 \qquad [03\text{-}03]$$

The following then generally applies for the kinetic energy at low velocities:

$$\left(\frac{1}{\sqrt{1-\frac{v_b{}^2}{c^2}}}-1\right)m_a c^2=\frac{m_a}{2}v_b{}^2$$

This then gives: $\frac{1}{\sqrt{1-\frac{v_b{}^2}{c^2}}}=\frac{v_b{}^2}{2c^2}+1$ [03-04]

The relativistic momentum conservation also applies between both masses:

$$\frac{m_{1a}}{\sqrt{1-\frac{v_{1b}{}^2}{c^2}}}v_{1b}=m_{1b}v_{1b}=\frac{m_{2a}}{\sqrt{1-\frac{v_{2b}{}^2}{c^2}}}v_{2b}=m_{2b}v_{2b}$$

The same series expansion for low velocities referred to above leads to the following expression with [03-04]:

$$m_{1a}v_{1b}+m_{1a}\frac{v_{1b}{}^3}{2c^2}=m_{2a}v_{2b}+m_{2a}\frac{v_{2b}{}^3}{2c^2}$$

Although in this case, the low velocities of both masses are considered in the third power, the terms can still be neglected because of the square of the speed of light in the denominator. Thus, we can write in a simplified way:

$$m_{1a}v_{1b}=m_{2a}v_{2b} \qquad [03\text{-}05]$$

It follows directly from the equation system [03-03] and [03-05] that:

$$v_{1b}=\sqrt{\frac{2\gamma m_{2a}}{\frac{m_{1a}}{m_{2a}}+1}\left(\frac{1}{r_1+r_2}-\frac{1}{r}\right)} \qquad \boxed{v_{1b}=\sqrt{\frac{2\gamma m_{2a}}{\frac{m_{1a}}{m_{2a}}+1}\,\frac{h}{r(r-h)}}} \qquad [03\text{-}06]$$

$$v_{2b}=\sqrt{\frac{2\gamma m_{1a}}{\frac{m_{2a}}{m_{1a}}+1}\left(\frac{1}{r_1+r_2}-\frac{1}{r}\right)} \qquad \boxed{v_{2b}=\sqrt{\frac{2\gamma m_{1a}}{\frac{m_{2a}}{m_{1a}}+1}\,\frac{h}{r(r-h)}}} \qquad [03\text{-}07]$$

It is obvious that the velocity of a very small mass m_2, moving towards a very large one (m_1), can be expressed simply as:

$$v_{2b}=\sqrt{2\gamma m_{1a}\,\frac{h}{r(r-h)}}$$

Based on the values in the legend of Fig. 03-01, the Earth has a top speed of 120 m/s, and the Moon 9,749 m/s in accordance with [03-07].

We will now consider the time dilation of the masses. The larger the mass (due to gravitational effect or velocity), the slower that time passes. As will later be shown in Chapter 7, there is symmetry between mass and time. Comparable relativistic formulas apply.

For example, Friedrich W. Seemann [9] has explained time dilation with the aid of the (impossible) change in frequency of an atomic clock in free fall in a gravitational field.

Using equations [03-01] and [03-02], it is possible to write:

$$\frac{\gamma m_{2a}}{c^2}\left(\frac{1}{r_1+r_2}-\frac{1}{r}\right)-\frac{\Delta m_2}{m_{1a}}=\frac{\Delta m_1}{m_{1a}} \qquad [03\text{-}08]$$

Employing equation [02-11] and taking into account the time dilation, the following applies:

$$\frac{\gamma m_{2a}}{c^2}\left(\frac{1}{r_1+r_2}-\frac{1}{r}\right)-\frac{\Delta m_2}{m_{1a}}=-\frac{\Delta t_1}{t_{1b}} \qquad [03\text{-}09]$$

Changed in accordance with the time dilation, this gives:

$$\Delta t_1=\left[\frac{\Delta m_2}{m_{1a}}-\frac{\gamma m_{2a}}{c^2}\frac{h}{r(r-h)}\right]t_{1b} \qquad [03\text{-}10]$$

The flow of time t_{1b} is inserted, corresponding to the derivation for equation [16-05]:

$$\Delta t_1=\left[\frac{\Delta m_2}{m_{1a}}-\frac{\gamma m_{2a}}{c^2}\frac{h}{r(r-h)}\right]\frac{2t_{1a}c^2}{2c^2+{v_{1b}}^2} \qquad [03\text{-}11]$$

The following can be written for the velocity:

$$\Delta t_1=\left[\frac{\Delta m_2}{m_{1a}}-\frac{\gamma m_{2a}}{c^2}\frac{h}{r(r-h)}\right]\frac{t_{1a}}{1+\frac{\gamma m_{2a}}{\left(\frac{m_{1a}}{m_{2a}}+1\right)c^2}\frac{h}{r(r-h)}} \qquad [03\text{-}12]$$

The expression Δm_2 needs to be replaced. Employing the conventions from [03-01], the following then applies:

$$E_{kin2}=\Delta m_2c^2=\frac{m_{2a}}{2}{v_{2b}}^2$$

This then gives the following expression for the dilation of time:

$$\Delta t_1=\left[\frac{{v_{2b}}^2 m_{2a}}{2\,c^2\,m_{1a}}-\frac{\gamma m_{2a}}{c^2}\frac{h}{r(r-h)}\right]\frac{t_{1a}}{1+\frac{\gamma m_{2a}}{\left(\frac{m_{1a}}{m_{2a}}+1\right)c^2}\frac{h}{r(r-h)}} \qquad [03\text{-}13]$$

[9] Friedrich W. Seemann, Was ist Zeit?, P. 98, P. 268, Wissenschaft und Technik Verlag Berlin, 2002, ISBN 3-89685-501-8

With the velocity of mass m_2 in accordance with [03-07], the following then applies:

$$\Delta t_1 = \left[\frac{\gamma}{\left(\frac{m_{2a}}{m_{1a}}+1\right) c^2} \frac{h}{r(r-h)} m_{2a} - \frac{\gamma m_{2a}}{c^2} \frac{h}{r(r-h)}\right] \frac{t_{1a}}{1+\frac{\gamma m_{2a}}{\left(\frac{m_{1a}}{m_{2a}}+1\right)c^2} \frac{h}{r(r-h)}} \qquad [03\text{-}14]$$

This expression can be simplified even further. The next item is the time dilation of mass m_1 on the surface of m_2, which occurs because of gravity, in contrast to the surface distance h between both masses and which can thus be calculated without the application of the general theory of relativity:

$$\boxed{\Delta t_1 = -\frac{t_{1a}}{\frac{c^2\, r(r-h)\, (m_{1a}+m_{2a})}{h\, {m_{2a}}^2\, \gamma}+1}} \qquad [03\text{-}15]$$

This is followed fully symmetrically for the time dilation of m_2, which occurs on the surface of the mass m_1, in comparison to a surface distance of h between the masses:

$$\boxed{\Delta t_2 = -\frac{t_{2a}}{\frac{c^2\, r(r-h)\, (m_{1a}+m_{2a})}{h\, {m_{1a}}^2\, \gamma}+1}} \qquad [03\text{-}16]$$

If a mass is accelerated to the centre of mass by gravitation, the gravitational and the velocity-related time dilations will be equal!

A mass m_2, located on the surface of m_1, experiences (in contrast to its stay at height level h) the same time dilation that would result by reaching its terminal velocity after falling from this height onto mass m_1.

If two masses are in a gravitational interaction with each other, a time dilation must also exist for each of the two masses. If one mass is very small in relation to the other (for example, a stone falling onto the Earth), the greater mass will, of course, hardly be affected at all. However, the principle of two corresponding systems does apply.

Let us suppose a spherical spaceship is 300,000 metres above the earth. It has a mass of 2000 kg and a radius of 2 m. How much time will have passed on the spaceship when one second elapses on Earth? (See the legend for Fig. 03-01).

Using equation [03-16], we get a time dilation of Δt_2 = -3.13 x 10^{-11} s. The value is negative, because it relates to the ground. This means that while one second passes on Earth, a total of 1 + (3.13 x 10^{-11}) seconds pass on the spaceship.[10] Using [03-15], we can calculate the time dilation Δt_1 of the Earth with respect to the spaceship. However, it is insignificantly small.

[10] A similar example, calculated using the Newtonian approximation of the gravitational potential, is provided at http://en.wikipedia.org/wiki/Time_dilation